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THE PRINCIPLES OF PURE MATHEMATICS.

Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten gradliniger Einheiten in elementarer Form entwickelt. Von Guiseppe Veronese, Professor an der königl. Universität zu Padua. xlvi. + 710 pp. (Leipzig: Teubner, 1894.)

THE work before us is an authorised German translation, by Lieutenant Adolf Schepp, of Wiesbaden, of Prof. Veronese's treatise on the Foundations of Geometry, first published at Padua in 1891; and the translator tells us in his short preface that the author has communicated to him the corrections and improvements of his work that have occurred to him since its publication in Italian. To give such an account of the contents of a book, so important, so original, and, it may be added, so controversial, as would serve to render its purpose and method generally intelligible, and at the same time to subject it to adequate criticism, would require a memoir rather than a review. We shall therefore endeavour only to give such a description as will recommend it to the attention of all who are interested in the logical basis of Pure Mathematics.

The Preface of thirty-four pages gives a general view of the author's system; the Introduction of 222 pages is devoted to the logical establishment of the notions of Number and Continuous Quantity; after that the First Part deals with the Straight Line, the Plane, and Space of Three Dimensions; the Second Part is occupied with the theory of Space of four or *n* Dimensions. The treatise ends with an interesting historical and critical discussion of the most important previous works on the same subject, and some notes elucidatory of special principles. A full table of contents is given, and a list of authors quoted; but it would have added much to the value of the book as a work of reference if a good alphabetical index of subjects had been included. The remainder of the present notice will be confined to the Introduction, the specially geometrical parts of the book being reserved for another occasion.

The thorough revision to which, in this century, the underlying principles of mathematical reasoning have been subjected is not less remarkable than the great advances made in the ulterior developments of these principles. Crelle has said that for those who probe the depths, equally with those who build in the heights of mathematical thought, there ever remain unexplored mysteries. Among those who have probed the depths, the investigators, namely, who have occupied themselves with the notions of number and quantity, the continuum of real numbers, the infinitely great and the infinitely small, there has been much divergence of opinion as to the logical grounding of the subject. Such discussions appear to be foreign to the taste of our English writers. With us Arithmetic is an affair of sums to be done by rules; Algebra is arrived at by noting the laws of operation with the numbers of Arithmetic, and giving to them the power of holding generally. We are too used to the process humorously described by Clifford [1]—"In the science of number while five-sevenths of fourteen has a meaning, namely, ten, five-sevenths of twelve is nonsense. Let us then treat it as if it were sense, and see what comes of it." This method, which Clifford held to be "logically false and educationally mischievous," is not that adopted by continental writers, and in particular it is not the method of Prof. Veronese. For him it is necessary so to define the abstract notions—number, quantity, and so on—that the laws of operation with them may be logically well-grounded upon the definitions. Let us see how he sets about the notion of (positive integral) number.

Readers of Clifford's lecture just now quoted will remember that the *crux* of Arithmetical theory lies in the proof of the statement that the number of things in a group is independent of the order in which they are counted; and the difficulty of proving it is not diminished by the facts, firstly, that everyone is firmly convinced of its truth, and secondly, that the whole system of Arithmetic is the work of the human mind, and therefore the theorem must somehow be implicitly included in the definitions when these are given with sufficient clearness. Prof. Veronese, like Kronecker,[1] appears to regard the ordinal number as logically preceding the cardinal number; in other words, he makes the idea of a group of things arranged in an order more elementary than the idea of the number obtained by counting the things in the group. Kronecker, going out from this notion, rapidly arrived at the required result, but our author is not satisfied with his reasoning. His own process is much more leisurely. He starts from the notions of unity and multiplicity (Einheit and Mehrheit) and explains the operations of uniting (Vereinigung) the objects of a series into a group, and of separation (Zerlegung) of a group into objects by successively taking away (Wegnehmen) object after object from the group. He defines an ordered group, and explains the unique correspondence of elements in two such groups. It is only after all this that he is prepared to define a number as an ordered group of units arranged to correspond uniquely and in the same order to the objects in an ordered group of objects. This definition is found to be a sufficient ground for the definition of counting, for proving the crucial proposition above referred to, and for the establishment of the commutative law of addition and the remaining laws of operation with positive integral numbers.

To establish the notion of Quantity and the extended conception of Number, with which Algebra has made us familiar, it is necessary, as many writers (including Du Bois Reymond) have pointed out, to frame an account of the Fundamental Form, or, as we may call it, the *Locus in quo* of real Quantity. A numerical fraction implies a something divisible into equal parts, and a part of it containing a certain number of these parts. A square root implies an exact measurement of a side of a square which has a given area. These examples show that we do tacitly or expressly assume a *somewhat* capable of exact division into parts in arbitrary ways corresponding to various mathematical ideas. This *somewhat* is the Fundamental Form (Grundform), and it has been frequently figured as a geometrical straight line, as by Du

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[1] "Lectures and Essays," vol. i. p. 356.

[1] "Ueber den Zahlbegriff." *Crelle.* Bd. ci. 1887.

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Bois Reymond (in his Algemenie Functionentheorie). According to Prof. Veronese it is more proper to give an independent abstract account of the Fundamental Form, partly because the properties of the straight line are afterwards to be determined in accordance with abstract definitions. He therefore takes as a guide the rectilinear continuum of intuition, of whose properties he gives an analysis, and then proceeds in an abstract manner. He defines a Form as anything whose marks are part, whole, order, and kind of position. Thus a line regarded as consisting of segments limited by points in a certain order having positions on the line is a Form, a song regarded as consisting of certain words pronounced in a certain order and each in a certain musical pitch is a Form. He explains how one Form may be determined by other Forms, and how the identity of two Forms may be inferred from the identity of the Forms that severally determine them. To avoid the circular reasoning that must ensue, unless some Forms are known to be identical there arises the necessity for introducing the Fundamental Form as a standard which serves for the determination of all others. He describes successively a system of one dimension as a form given by a series of elements whose order, from a certain element, is a mark of the form, a homogeneous system of one dimension, and a system of one dimension identical in the positions of its parts. Such a form is chosen as Fundamental Form. The operations of uniting segments of the Form and of separating united segments are described, and shown to obey the Laws of Algebra for addition and subtraction. The relations of segments as multiples or factors of other segments lead to the laws of multiplication and division, and to the description of the Scale founded upon any segment as unit. The Range of the Scale (Gebiet der Scala) is the part of the Fundamental Form arrived at by continual repetition of the segment chosen as unit.

The author is now prepared to introduce the conceptions of the infinitely great and the infinitely small. He assumes that there is an element of the Fundamental Form which lies outside the Range of the Scale founded on any segment as unit. This assumption is apparently free from any contradiction. Such an element being chosen, the segment limited by it, and any element within the range of the scale, is infinitely great in reference to the unit of the scale; had this segment been chosen as unit, the original unit would have been infinitely small. From the nature of the Fundamental Form, as a homogeneous system identical in the position of its parts, follows the necessity of assuming any number of orders of infinite segments and any number of orders of infinitesimal segments.

To every segment corresponds a numerical symbol, just as in particular the natural numbers correspond to the segments which are exact multiples of that one chosen as unit. The ordinary Laws of Algebra holding for the segments hold in like manner for the numbers thus introduced. To the infinitely great and infinitely small segments of different orders correspond infinitely great and infinitely small numbers of different orders. It is proved that the numbers thus arrived at are not identical with Cantor's "Transfinite numbers" (*Acta Mathematica*, Bd. II.). After the introduction of these numbers, and the establishment of the laws of operation with them, come the hypotheses of continuity of the Fundamental Form, an idea here treated in a very instructive manner, the proof of the existence of Limits so elaborately discussed by Du Bois Reymond, the notions of commensurable and incommensurable segments, and the theory of Proportion, the last being especially interesting. A chapter is added for the sake of completeness, in which the properties of real, positive and negative, rational and irrational, numbers are established on the basis of principles already discussed.

It is a cardinal feature of the author's account of the theory of Quantity to dispense with the so-called Axiom V of Archimedes, according to which it is inherent in the notion of quantity that when one quantity a is greater than another b, there exists a number n such that nb is greater than a. This axiom has been found by other writers, as Stolz, extremely useful in establishing the properties and relations of finite quantities, but appears to involve difficulties in connection with the infinitely great and the infinitely small. At the expense of greater length of explanation, Prof. Veronese has freed the theory from the axiom and the involved difficulties. His own exposition is generally clear, though his doctrine of "commensurable numbers of the second kind" (pp. 182 and 213) is not without obscurity. Could not an example have been given?

Enough has been said to show that Prof. Veronese's book treats of a great deal besides the Foundations of Geometry—his Introduction might, in fact, well be entitled the Foundations of Mathematics. He tells us that although some parts of it will be useful in Geometry, much has been worked out simply for its own sake. We may well be grateful to him for the patience and trouble that he has expended in clearing up the Logic of the operations that most of us, without a thought of underlying difficulties, cheerfully perform with confidence and success. He has none of the charm of style to be found in the writings of Clifford or Kronecker. Rigorous he is, thoroughly common-sense, careful almost to tediousness, and extremely leisurely. For the elucidation of the very difficult subject he has chosen, these qualities are perhaps the greatest of merits, yet we fear that they will not render his writings acceptable to readers unprepared for a considerable sacrifice of time. A. E. H. L.

TEXT-BOOKS ON ORGANIC CHEMISTRY.

Organic Chemistry. Part I. By W. H. Perkin, jun., Ph.D., F.R.S., and F. Stanley Kipping, Ph.D., D.Sc. (London: W. and R. Chambers, 1894.)

Lessons in Organic Chemistry. Part I. Elementary. By G. S. Turpin, M.A. (Camb.), D.Sc. (Lond.) (London: Macmillan, 1894.)

IT is not surprising that "organic" chemistry should have received less attention in this country than on the continent, considering that the professors in nearly all the chief British universities have been notoriously neglectful of this department of the science, and that the highest degrees in connection with chemical science have been until recent years generally attainable without a practical acquaintance with the subject, and without evidence of capacity for research.

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